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Bubbling AdS$_3$

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Abstract: In the light of the recent Lin, Lunin, Maldacena (LLM) results, we investigate 1/2-BPS geometries in minimal (and next to minimal) supergravity in $D = 6$ dimensions. In the case of minimal supergravity, solutions are given by fibrations of a two-torus T^2 specified by two harmonic functions. For a rectangular torus the two functions are related by a non-linear equation with rare solutions: $AdS_3 \times S^3$, the pp-wave and the multi-center string. "Bubbling", i.e. superpositions of droplets, is accommodated by allowing the complex structure of the T^2 to vary over the base. The analysis is repeated in the presence of a tensor multiplet and similar conclusions are reached, with generic solutions describing D1D5 (or their dual fundamental string-momentum) systems. In this framework, the profile of the dual fundamental string-momentum system is identified with the boundaries of the droplets in a two-dimensional plane.

Keywords: D-branes, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

The AdS/CFT correspondence [1] relates deformations of AdS geometries to states in the boundary CFT. A concrete realization of this idea was recently proposed by Lin, Lunin, Maldacena (LLM) in [2], where classical geometries of Type IIB supergravity corresponding to 1/2-BPS states in $\mathcal{N} = 4$ SYM were constructed. On the gauge theory side the solutions correspond to chiral primary operators with conformal weight $\Delta = J$ and are dual to deformations of $AdS_5 \times S^5$ (or the pp-wave) backgrounds preserving half supersymmetry. The field theory states were previously found to have a description in terms of free fermions [3, 4]. In a semiclassical limit these states can be depicted as "droplets", or "bubbles", on a two-dimensional plane, which is the phase space of these fermions. The remarkable result of [2] was to show that to droplet configurations correspond supersymmetric solutions of Type IIB supergravity, with SO(4) $\times$ SO(4) isometry. The geometries are completely specified by a distribution of charge $z = \pm 1/2$ on a two dimensional plane non-trivially embedded in space-time. In the geometry the "bubbles" define islands in space where one of the two S^3 shrinks to zero size and only for $z = \pm 1/2$ the corresponding geometries are regular.

The results in [2] provide the most general supersymmetric solutions of Type IIB supergravity, in the presence of a five-form flux admitting an SO(4) $\times$ SO(4) group of isometries.[1] This analysis was done using the techniques first introduced in [6] (and subsequently exploited in [7]–[10] and many others) and is greatly simplified by the large amount of isometry. The field theory duals are given by SYM states satisfying $\Delta = J$ built as multi-trace products of J scalar fields of a single specie.

[1] In [2] a similar analysis was applied to M-Theory geometries, and constituted an extension of the results of [5].

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The scope of this note is to investigate, from the supergravity point of view, the similar story in $D = 6$. Solutions will correspond to 1/2-BPS deformations of $AdS_3 \times S^3$ (or its pp-wave limit) and are dual to chiral primaries in the boundary CFT. The existence of a free fermion description of primaries in the two-dimensional CFT [11] suggests that bubbling solutions should find room in six-dimensional supergravity. Here we show that this is indeed the case. Much is already known about the supergravity description of chiral primaries of the two-dimensional CFT [12, 13, 14], and we will ask ourselves whether these results can be reinterpreted in terms of bubblings of AdS_3.

Half BPS geometries associated to excitations around $AdS_3 \times S^3$ are dual to chiral primaries in the two-dimensional D1D5 (or FNS) CFT. The spectrum of chiral primaries and its dual KK descendants in supergravity have been worked out in [15]–[18]. States in the CFT are classified by four charges $h, \bar{h}, j, \bar{j}$ describing the quantum numbers under the isometry group $\mathrm{SO}(2,2) \times \mathrm{SO}(4) \sim \mathrm{SL}(2,R)_L \times \mathrm{SL}(2,R)_R \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. $h, \bar{h}$ describe the conformal dimension of the two-dimensional CFT and $j, \bar{j}$ the R-symmetry charges. In the case of minimal $\mathcal{N} = (1,0)$ supergravity in $D = 6$, the CFT has $\mathcal{N} = (4,0)$ supersymmetry. This sector is universal to any supergravity in $D = 6$ and solutions are shared by supergravities following from reductions on T^4, $K3$ and orientifolds.

In analogy with the ten-dimensional case we start by decomposing the isometry group as $\mathrm{SO}(2)^2 \times \mathrm{SO}(2)^2_{\theta_1,\theta_2}$ and consider states with zero $\mathrm{SO}(2)^2_{\theta_i}$ charges i.e. $h = \bar{h}$ and $j = \bar{j}$. 1/2-BPS states correspond to chiral primaries $h = j$ and therefore we look for states with $h = \bar{h} = j = \bar{j} = m/2$. There is a single state of this type for each m in the spectrum of KK descendants of the gravity multiplet and one for each tensor multiplet.[2] We therefore look for solutions in the pure $\mathcal{N} = (1,0)$ supergravity and its minimal extension by adding a tensor multiplet.

Notice that, contrary to the ten-dimensional case studied in [2], requiring the solution to admit an $\mathrm{SO}(2) \times \mathrm{SO}(2)$ group of isometries *does not* fix uniquely the form of the internal space. In particular, on the two-torus T^2 we could have a non-diagonal metric, and generically a non-trivial fibration structure. We start our analysis being conservative, working in the minimal supergravity, and mimicking the ansatz used in [2] with $T^2 = S^1 \times S^1$. Rather surprisingly, we find an almost identical set of equations describing our solutions. In particular, it turns out that a function z obeys the same equation as in [2]. However, unlike in the LLM case, the Bianchi identity translates into a further harmonic condition on the function h^2, related via a non-linear equation to z. The important property of linearity of the equations is in this way lost and solutions are rare: $AdS_3 \times S^3$, the pp-wave and the multi-center string.

It turns out that the resolution of this problem arises from relaxing the initial metric ansatz, namely, considering a torus which is not any more rectangular. Indeed, using the more general form of 1/2-BPS solutions of minimal supergravity given in [10], we show then how bubbling can be accommodated by allowing for a tilted T^2. In this case, the non-linear relation between z and h^2 is lifted, and one is able to freely superpose different solutions in

[2]This can be easily seen from the list (3.1) in [17, 18] for KK descendants of the various $\mathcal{N} = (1,0)$ supermultiplets.

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a fashion similar to [2]. The resulting geometries are given in terms of harmonics generated by lines of charges distributed along the boundary of droplets in a two-dimensional plane. The cycles of the torus degenerate along this plane, while crossing the charged strings the corresponding pinching cycles get flipped.

Finally we extend our analysis by adding a tensor multiplet to the minimal theory, namely an anti-self-dual three-form, and a scalar field. This theory includes a wider class of D1D5 classical geometries like for instance giant gravitons, and have been systematically studied in [13]. The familiar string profiles describing these solutions are reinterpreted here as the boundaries of the droplet configurations in the two-dimensional plane.

The paper is organized as follows: In section 2 we describe the solutions for minimal supergravity in $D = 6$. We start by considering a simple metric ansatz where the torus of isometries is rectangular. In section 2.3 the solutions are written in the canonical form of [10] and the ansatz for the metric is relaxed to accommodate bubblings. In section 2.4 we discuss the general features of bubbling solutions. In section 3 we add a tensor multiplet and discuss bubblings in this extended framework. Finally, in 4 we draw some conclusions.

Note added: while this work was being completed, the paper [19] appeared, which overlaps with the results in our section 2.2.

2. 1/2-BPS solutions in minimal supergravity

2.1 The supersymmetry conditions

In this section we make use of the results of [10] to find 1/2-supersymmetric solutions of minimal $\mathcal{N} = (1, 0)$ supergravity in 6 dimensions of the type recently constructed in [2].[3] We use the six-dimensional conventions of [10], and adhere to the notation of [2].

Minimal supergravity in 6 dimensions comprises a graviton g_{mn}, a two-form B^+_{mn} with self-dual field strength, and a symplectic Majorana-Weyl gravitino ψ^A_μ. The Killing spinor equation reads:

$$\nabla_m \epsilon - \frac{1}{4} G_{mnp} \gamma^{np} \epsilon = 0 \tag{2.1}$$

where $G = dB^+$ is self-dual and ϵ is symplectic-Majorana-Weyl, i.e. it has 8 real degrees of freedom.

All supersymmetric solutions of minimal supergravity were characterized in [10] in terms of a vector V and a triplet of self-dual three-forms X^i constructed in terms of spinor bi-linears. These objects satisfy some algebraic constraints, following from Fierz identities, and some differential conditions which are equivalent to the supersymmetry equation (2.1). For instance, one of the algebraic constraints implies that the vector V is null

$$V_m V^m = 0 \,. \tag{2.2}$$

The Killing spinor equation (2.1) is then equivalent to the following differential conditions

$$\nabla_m V_n = V^p G_{pmn} \tag{2.3}$$

[3]The results in this section were derived in collaboration with G. Dall'Agata.

$$\nabla_m X^i_{npq} = G_{mp}{}^r X^i_{rqn} + G_{mq}{}^r X^i_{rnp} + G_{mq}{}^r X^i_{rpn} \,. \tag{2.4}$$

Notice that (2.3) implies that V is a Killing vector. We refer to [10] for further details concerning the algebraic and differential conditions.

In analogy with [2], we now introduce ansatze for the metric and the self-dual 3-form that admit a $\mathrm{U}(1)_{\theta_1} \times \mathrm{U}(1)_{\theta_2}$ group of isometries

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, dx^\mu dx^\nu - e^{H+G}\, d\theta_1^2 - e^{H-G}\, d\theta_2^2 \\ G &= F \wedge d\theta_1 + \tilde{F} \wedge d\theta_2 \end{aligned} \tag{2.5}$$

with $\mu = 0, 1, 2, 3$. Self-duality of the three-form[4] field strength G yields the following relations:

$$F \equiv dB = e^G \ast_4 \tilde{F} \qquad \tilde{F} \equiv dB = -e^{-G} \ast_4 F \tag{2.6}$$

where $\ast_4$ is the Hodge star operator with respect to the metric $g_{\mu\nu}$. We would like now to use the constraints imposed by supersymmetry in order to determine the functions entering in the metric as well as the two-form F in (2.5). This can be easily achieved using the results of [10]. Although in this latter reference it was introduced a null orthonormal frame adapted to the Killing vector V, we find it more convenient to adapt the conditions in [10] to our metric ansatz. In particular, we will first extract from the six-dimensional bi-linears a set of four-dimensional forms.

The null Killing vector V^m can be always chosen of the form $V^m = (K^\mu, f_1, f_2)$. Since $\nabla_{(m}V_{n)} = 0$, we also have that K^μ is a Killing vector, as well as that $\partial_\mu f_1 = \partial_\mu f_2 = 0$. We can therefore normalize our vector so that

$$V^m = (K^\mu, 1, 1)\,, \qquad V_m = (K_\mu, -e^{H+G}, -e^{H-G})\,. \tag{2.7}$$

As V is null, it follows that the vector K is timelike, and its norm is given by

$$K \cdot K = e^{H+G} + e^{H-G} \equiv h^{-2}\,, \tag{2.8}$$

hence we can choose a time coordinate t such that the metric takes the form

$$ds^2 = h^{-2}(dt + C)^2 - g^3_{ij} dx^i dx^j - e^{H+G}\, d\theta_1^2 - e^{H-G}\, d\theta_2^2 \tag{2.9}$$

whence

$$K = \frac{\partial}{\partial t} \quad \text{as a vector} \tag{2.10}$$

$$K = h^{-2}(dt + C) \quad \text{as a one} - \text{form} \tag{2.11}$$

$C = C_i\, dx^i$, $i = 1, 2, 3$ and of course nothing depends on t, θ_1 or θ_2. To proceed, we define a set of forms by decomposing the 3-forms X^i

$$X^i_{\mu\theta_1\theta_2} = L^i_\mu\,, \quad X^i_{\mu\nu\theta_1} = Y^i_{\mu\nu}\,, \quad X^i_{\mu\nu\theta_2} = \tilde{Y}^i_{\mu\nu}\,, \quad X^i_{\mu\nu\rho} = \tilde{L}^i_{\mu\nu\rho}\,. \tag{2.12}$$

Due to the self-duality of X^i, these are not all independent (see appendix A for details). It turns out that the differential equations that these forms satisfy determine the complete form of the metric and self-dual three-form G. We have relegated the detailed derivation in the appendix A.

[4]Hopefully it should be clear when G denotes the three-form and when it denotes a function.

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2.2 General solution for a rectangular torus

The final result is the following. The metric is specified by a single function G and is given by:

$$\begin{aligned} ds^2 &= h^{-2}(dt+C)^2 - h^2(dy^2+dx_1^2+dx_2^2) - ye^G\,d\theta_1^2 - ye^{-G}\,d\theta_2^2 && (2.13)\\ h^{-2} &= 2y\cosh G && (2.14)\\ dC &= \frac{1}{y}*_3\,dz \qquad z=\frac{1}{2}\tanh G && (2.15)\end{aligned}$$

where $*_3$ is the Hodge star in the flat metric $ds_3^2 = dy^2 + dx_1^2 + dx_2^2$. Notice that z and h^2 are defined *exactly* as in [2]. The three-form is given by

$$\begin{aligned} F &= dB_t \wedge (dt+C) + B_t dC + d\hat{B}\\ F &= d\tilde{B}_t \wedge (dt+C) + \tilde{B}_t dC + d\hat{\tilde{B}}\\ B_t &= \frac{1}{2}ye^G \qquad \tilde{B}_t = \frac{1}{2}ye^{-G}\\ d\hat{B} &= -d\hat{\tilde{B}} = \frac{1}{2}y\ *_3\,dh^2\,. && (2.16)\end{aligned}$$

Consistency of equations (2.15) and (2.16) requires that $d(dC)=0$ and $d(d\hat{B})=0$. These impose *two* second order equations to be satisfied by h^2 and z:

$$\Delta_3 z - \frac{1}{y}\partial_y z = 0 \qquad (2.17)$$

$$\Delta_3 h^2 + \frac{1}{y}\partial_y h^2 = 0 \qquad (2.18)$$

with $\Delta_3 = \partial_y^2 + \partial_1^2 + \partial_2^2$. Recalling that h and z are related by the equation

$$h^2 = \frac{1}{y}\sqrt{\frac{1}{4}-z^2} \qquad (2.19)$$

we see that eqs. (2.17) and (2.18) are two differential equations on a single function z. This is substantially different from [2] where the equations $d\hat{B}=0$ and $d\hat{\tilde{B}}=0$ were automatically satisfied for a function z obeying (2.17).

Notice that like in [2], solutions are regular for $z=\pm 1/2$ on the two dimensional plane $y=0$. Indeed, only for these values of z, the shrinking S^1 at $y=0$ combine with y to reconstruct a regular (i.e. free of conical singularities) $\mathbb{R}^2\times S^1$. Therefore, exactly like for LLM, non-singular solutions are completely specified by two-dimensional figures (on the $y=0$ plane) representing regions where $z=\pm\frac{1}{2}$. However, solutions to (2.17)–(2.19) are now sporadic. Remarkably, we find that $AdS_3\times S^3$, the pp-wave and the multi-center string do satisfy these equations. They correspond to the simplest figures: the disk, the upper half-plane and points (or small far away droplets). We will refer to the filled figures as droplets. The functions z and h^2 specifying these solutions were given in [2] and will be displayed below momentarily.

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Equations (2.17) and (2.18) can be expressed as laplacian equations in $d = 6$ and $d = 4$

$$\Delta_6 \left(\frac{z}{y^2}\right) = 0 \tag{2.20}$$
$$\Delta_4 \, h^2 = 0 \tag{2.21}$$

where y is interpreted as the radius of the extra S^3 and S^1 auxiliary spheres respectively. The function z and the one-form C can be written in terms of integrals over the boundary of the corresponding droplets in the $y = 0$ plane [2]:

$$z(x_1, x_2, y) = \frac{y^2}{\pi} \int_{\mathcal{D}} \frac{z(x_1', x_2', 0)\, dx_1' dx_2'}{(|\mathbf{x} - \mathbf{x}'|^2 + y^2)^2} = \sigma - \frac{1}{2\pi} \int_{\partial\mathcal{D}} dv |\partial_v \mathbf{x}'| \, \frac{\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}'(v))}{|\mathbf{x} - \mathbf{x}'(v)|^2 + y^2} \tag{2.22}$$
$$C_i(x_1, x_2, y) = \frac{\epsilon_{ij}}{\pi} \int_{\mathcal{D}} \frac{z(x_1', x_2', 0)\, (x_i - x_i')\, dx_1' dx_2'}{(|\mathbf{x} - \mathbf{x}'|^2 + y^2)^2} = \frac{\epsilon_{ij}}{2\pi} \int_{\partial\mathcal{D}} dv \frac{\partial_v x_j'(v)}{|\mathbf{x} - \mathbf{x}'(v)|^2 + y^2} \tag{2.23}$$

In the right hand side of the equations we have introduced a parametrization of the one-dimensional boundary $\partial\mathcal{D}$ of the droplets in the $y' = 0$ plane. From (2.23) we see that C_i are harmonic functions in $d = 4$ generated by a one-dimensional charge distribution along the line $(x_1'(v), x_2'(v))$ parametrized by v, with charge density $\partial_v x_j'(v)$. $\sigma = \pm\frac{1}{2}$ is a contribution coming from infinity arising for solutions for which $z = \pm\frac{1}{2}$ outside some circle of large radius [2].

Let us now consider the function h^2. Since this is a harmonic function in $d = 4$, the general solution is specified by some source distribution ρ, and can be written as

$$h^2(x_1, x_2, y) = \int_{\mathcal{D}} \frac{\rho(x_1', x_2')\, dx_1' dx_2'}{|\mathbf{x} - \mathbf{x}'|^2 + y^2} \,. \tag{2.24}$$

Note that this expression takes into account the fact that h^2 must be invariant along the auxiliary S^1, so that the density $\rho(x_1', x_2')$ should sit on the $y' = 0$ plane. The physical reason for this will become clearer later. Interestingly, this density can be computed explicitly in the three cases at hand where h^2 is related to z via the non-linear relation (2.19). It turns out that in all these cases we can write h^2 as the following boundary integral

$$h^2(x_1, x_2, y) = \frac{1}{2\pi} \int_{\partial\mathcal{D}} \frac{dv}{|\mathbf{x} - \mathbf{x}'(v)|^2 + y^2} \,. \tag{2.25}$$

This will be shown explicitly below for the basic figures: the circle, half-plane and points. One can wonder whether formulas (2.22), (2.25) can be extended to more complicated figures. The problem is that, if one tries to do so, the resulting solutions will fail to obey (2.19). In the next section we will show how this can be solved by relaxing the metric ansatz.

Finally, it is interesting to notice that it is also possible to trade the linear equation for h^2 in favor of a non-linear one for z, which reads

$$(\nabla z)^2 = \frac{4}{y^2} \left(\frac{1}{4} - z^2\right)^2 . \tag{2.26}$$

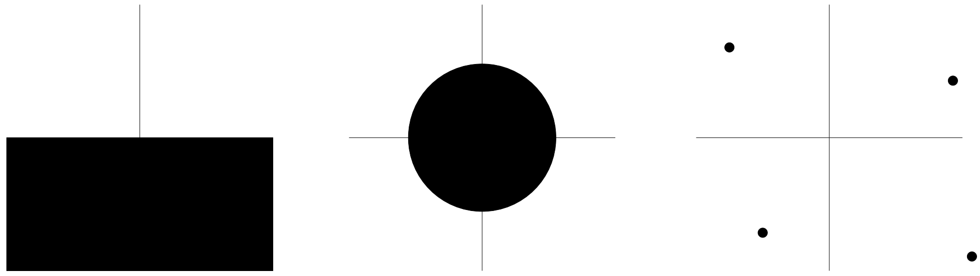

Figure 1: The basic figures in minimal supergravity. (*a*) The pp-wave. (*b*) $AdS_3 \times S^3$. (*c*) Multi-center string.

Examples: here we collect the form of z and h^2 for the simplest solutions (the only known to us) to the system (2.17)–(2.19).

pp-wave:

$$\begin{aligned} z &= \frac{1}{2}\frac{x_2}{\sqrt{x_2^2+y^2}} \\ h^2 &= \frac{1}{2}\frac{1}{\sqrt{x_2^2+y^2}}\,. \end{aligned} \tag{2.27}$$

This corresponds to dividing the $y=0$ plane in two regions (filled and empty), separated by the x_1 axis [2]. The functions z and h^2 can be written as the integrals (2.22), (2.25) over the x_1-axis dividing the two regions:

$$\mathbf{x}'(v) = (v, 0) \qquad -\infty < v < \infty\,. \tag{2.28}$$

$AdS_3 \times S^3$**:**

$$\begin{aligned} z &= \frac{1}{2}\frac{\mathbf{x}^2+y^2-a^2}{\sqrt{(\mathbf{x}^2+y^2+a^2)^2-4\,a^2\,\mathbf{x}^2}} \\ h^2 &= \frac{a}{\sqrt{(\mathbf{x}^2+y^2)^2-2a^2(\mathbf{x}^2-y^2)+a^4}}\,. \end{aligned} \tag{2.29}$$

This corresponds to a round disk of radius a centered in the origin [2]. The functions z and h^2 can be written as the integrals over the droplet boundary (a circle of radius a) parametrized by v:

$$\mathbf{x}'(v) = (a\cos\,v, a\sin\,v) \qquad 0 < v < 2\,\pi\,. \tag{2.30}$$

Multi-center string:

$$\begin{aligned} h^2 &= \frac{1}{\lambda^2}\,H \qquad H = \sum_i \frac{Q_i}{(\vec{x}-\vec{x}_{0,i})^2+y^2} \\ z &= \pm\sqrt{\frac{1}{4}-\frac{1}{\lambda^4}\,H^2y^2} \qquad \lambda \to \infty\,. \end{aligned} \tag{2.31}$$

In this case the equation for z is satisfied in the limit $\lambda \to \infty$. Suppose we take the plus sign in (2.31), then in the limit, $e^{-G} \sim yH/\lambda^2$. Substituting these into the metric one finds $C \approx 0$ and

$$ds^2 = H^{-1}(d\tilde{t}^2 - dw^2) - H(d\vec{\tilde{x}}^2 + d\tilde{y}^2 + \tilde{y}^2 d\theta_2^2) \tag{2.32}$$

where we have rescaled $\tilde{t} = \lambda\, t$, $w = \lambda\, \theta_1, \tilde{y} = \lambda^{-1}\, y$, $\vec{\tilde{x}} = \lambda^{-1}\, \vec{x}$. The resulting solution corresponds to a multi-center string in $D = 6$. Obviously, the same solution is obtained choosing the minus sign in (2.31), upon exchanging θ_1 and θ_2.

Notice that also in this case, the harmonic function h^2 can be thought of as arising from a profile, but now the boundary of the droplets are points $\partial\mathcal{D} = \{\mathbf{x_{0,i}}\}$. The profile function reads:

$$\mathbf{x}'(v_i) = \mathbf{x_{0,i}} \qquad 0 < v_i < Q_i\,.$$

2.3 Solutions in GMR form

Given that our geometries are supersymmetric solutions of minimal supergravity in six dimensions, there must be a change of coordinates that cast them in the general form presented in [10]. In the present section we give the map between the solutions in 2.2 and the canonical form of [10]. As a bonus, we will show how the condition (2.19) can be lifted by relaxing the metric ansatz for the torus of isometries allowing for an off-diagonal term.

Recall that the full six-dimensional metric can be always written [10] as

$$ds^2 = 2H^{-1}\left(du + \beta_m dx^m\right)\left(dv + \omega_m dx^m + \frac{\mathcal{F}}{2}\left(du + \beta_m dx^m\right)\right) - H h_{mn} dx^m dx^n \tag{2.33}$$

with the functions $H, \mathcal{F}$, the one-forms β, ω, and the four-metric h_{mn} obeying certain coupled differential equations. It turns out that the solutions of the previous section fall into the u-independent class of that considered in [10]. More precisely, out of the two null directions, we can define a time coordinate t and a coordinate α via

$$u = \frac{1}{\sqrt{2}}(t - \alpha) \qquad v = \frac{1}{\sqrt{2}}(t + \alpha)\,. \tag{2.34}$$

In addition we take:

$$\begin{aligned} \mathcal{F} &= 0 & H &= h^2 \\ \beta &= \frac{1}{\sqrt{2}}(C - z d\phi) & \omega &= \frac{1}{\sqrt{2}}(C + z d\phi)\,. \end{aligned} \tag{2.35}$$

The four-metric h_{mn} has to be hyper-Kähler [10], and we take this to be flat, of the form

$$h_{mn} dx^m dx^n = dx_1^2 + dx_2^2 + dy^2 + y^2 d\phi^2\,. \tag{2.36}$$

This is rather natural, given that we are interested in geometries dual to D1D5 systems (without momentum) moving in a flat transverse space. More general hyper-Kähler spaces are relevant for D1 states on a curved manifold (K3) or D1D5 systems with momentum [20, 21]. Note that the definition of ϕ, has been chosen such that it has periodicity 2π. In the new coordinates the metric can be written in the following form

$$ds^2 = h^{-2}\left[(dt + C)^2 - (d\alpha + z d\phi)^2\right] - h^2(dx_1^2 + dx_2^2 + dy^2 + y^2 d\phi^2) \tag{2.37}$$

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$$= h^{-2}(dt+C)^2 - h^2(dy^2+dx_1^2+dx_2^2) - \left[h^2y^2 + h^{-2}(z+\frac{1}{2})^2\right] d\theta_1^2$$

$$- \left[h^2y^2 + h^{-2}(z-\frac{1}{2})^2\right] d\theta_1^2 - 2\left[h^{-2}(\frac{1}{4}-z^2) - h^2y^2\right] d\theta_1\, d\theta_2 \qquad (2.38)$$

where in the second line we performed the change of variables

$$\theta_1 = \alpha + \frac{1}{2}\phi \qquad \theta_2 = \alpha - \frac{1}{2}\phi\,. \qquad (2.39)$$

Note that the $g_{\theta_1\theta_2}$ term in (2.38) precisely cancels using the fact that z and h^2 are related via (2.19), and the remaining terms recombine to reconstruct the metric (2.13).

It is easy to check that the equations of [10] are satisfied. Note in fact that

$$d\beta = \frac{1}{\sqrt{2}}(dC - dz\wedge d\phi) \qquad (2.40)$$

$$d\omega = \frac{1}{\sqrt{2}}(dC + dz\wedge d\phi) \qquad (2.41)$$

and that

$$(d\beta)^- = (d\omega)^+ = 0 \qquad (2.42)$$

is equivalent to equations (5.27), (5.30) of that paper. In particular these equations imply

$$dC = \frac{1}{y} *_3 dz \quad \Rightarrow \quad \Delta_6\left(\frac{z}{y^2}\right) = 0\,.$$

Here $(\ )^{\pm}$ indicates (anti)-self-dual part with respect to the four dimensional metric (2.36). The condition $(d\omega)^+ = 0$ of course implies that $\mathcal{G}^+ = 0$ as required by $\mathcal{F} = 0$. The Bianchi identity and Einstein equation reduce correctly to

$$\Delta_4 h^2 = 0\,. \qquad (2.43)$$

This constitutes a check on our solutions, as well as a proof that they indeed satisfy the Einstein equations. Moreover, this explains the reason why h^2 was previously found to be harmonic in *four* dimensions.

Finally, it can be checked that the expression for the flux in [10] (in the u-independent class):

$$G = \frac{1}{2} *_4 dh^2 - e^+ \wedge \frac{1}{2}d\omega + \frac{1}{2}h^{-2}e^- \wedge d\beta - \frac{1}{2}e^+\wedge e^- \wedge h^{-2}dh^2\,. \qquad (2.44)$$

agrees precisely with the expression for the flux in (2.16). For this, it is useful to note

$$e^- = \frac{h^2}{\sqrt{2}}\left[h^{-2}(dt+C) + e^{H+G}d\theta_1 + e^{H-G}d\theta_2\right] \qquad (2.45)$$

$$e^+ = \frac{1}{\sqrt{2}}\left[h^{-2}(dt+C) - e^{H+G}d\theta_1 - e^{H-G}d\theta_2\right]. \qquad (2.46)$$

To summarize, we have found that our solutions comprise a restricted sub-class of the u-independent solutions of [10]. This analysis is useful to understand how one can generalize the starting ansatz, in order to obtain more interesting solutions.

2.4 Minimal bubbling

The point to notice is that the metric (2.38) and 3-flux (2.44) are 1/2-BPS solution of minimal supergravity for any choice of the functions h^2 and z satisfying

$$\begin{aligned} dC &= \frac{1}{y} *_3 dz \\ \Delta_4 h^2 &= 0 \\ \Delta_6 \left(\frac{z}{y^2}\right) &= 0\,, \end{aligned} \tag{2.47}$$

however, crucially, one can now relax the requirement that z and h^2 be related as in (2.19). This suggests how to recover linearity: we lift the non-linear relation (2.19) and consider h^2 as an independent function with respect to z. According to (2.38), this results into a non-trivial $g_{\theta_1\theta_2}$ component in the metric. Note that in the case studied in [2], requiring the solution to posses an $SO(4) \times SO(4)$ isometry, uniquely fixed the factorized form of the metric in the internal six dimensional space. In the case at hand, the presence of an abelian $SO(2) \times SO(2)$ isometry allows us to retain the same isometry for a generic tilted T^2-torus.[5]

Motivated by the way of writing C_i and h^2 for the basic figures as the integrals (2.22) and (2.25), it is tempting to define "bubbling" of AdS_3 by extending these integrals to the boundary of a generic droplet distribution in the $y = 0$ plane. This intuition will be confirmed in the next section where we will show that solutions derived in this way agree with those found in [13, 14] for D1D5 classical geometries. These are clearly supersymmetric solutions, and as shown in [14], they are also non-singular.

We thus define a "minimal bubbling" as superposition of solutions of minimal supergravity, with basic building blocks the circular droplets (AdS_3) and the half planes (pp-wave) and $z = \pm 1/2$ over the plane.[6] Notice that these figures satisfy the condition $|\partial_v \mathbf{x}'(v)|^2$ =const. that ensures that we stay in minimal supergravity [20].[7]

Resuming, a bubbling solution of minimal supergravity is specified by a droplet distribution in the $y = 0$ plane. A generic droplet distribution in the $y = 0$ plane is represented in figure 2. The functions z and h^2 are given via the line integrals (2.22) and (2.25) over the boundary $\partial\mathcal{D}$ of the filled regions. The harmonic conditions are then automatically satisfied, and the solutions are non-singular.

As an illustration let us consider the annulus diagram[8] in figure 3.

The corresponding profile reads

$$\mathbf{x}'_i(v_i) = \left(a_i \cos\frac{v_i}{\xi_i}, a_i \sin\frac{v_i}{\xi_i}\right) \qquad 0 < v_i < 2\,\pi\,\xi_i \tag{2.48}$$

[5] One could consider the case in which this torus is non-trivially fibered over the external four-dimensional space, and allow more general terms like $g_{x^i\theta_j}$, but this goes beyond the scope of this note.

[6] The study of solutions with conical singularities is also interesting in $D = 6$ since they correspond to supergravity duals of chiral primaries in the twisted sector of the orbifold CFT [14]. They can be accommodated in the bubbling picture by taking a string profile covering m times the circle.

[7] Here we are using our identification of the D1D5 profiles with the boundary of the droplet configuration.

[8] An equally simple example is provided by the strip in figure 3.

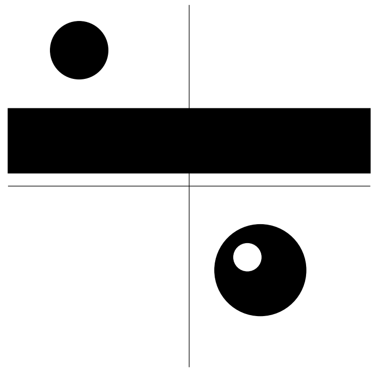

Figure 2: A generic distribution of droplets in minimal supergravity.

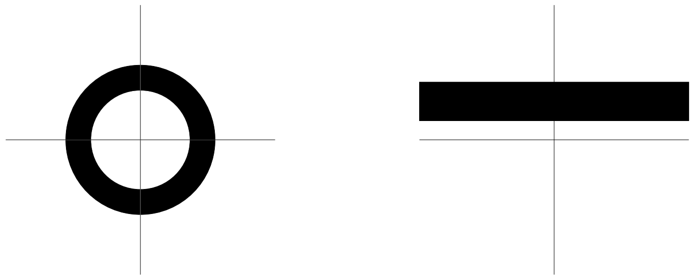

Figure 3: The annulus (or ring) and the strip.

with $i = 1, 2$, and $\xi_1 + \xi_2 = 1$. To stay in minimal supergravity we require $|\partial_v \mathbf{x}'(v)|^2 = a^2$, with a constant, i.e.

$$\frac{a_1}{\xi_1} = \frac{a_2}{\xi_2} = a\,.$$

Then the integrals (2.22), (2.23) and (2.25) result into:

$$\begin{aligned} z_{\rm ring}(\mathbf{x}, y) &= z(\mathbf{x}, y; a_1) - z(\mathbf{x}, y; a_2) + \frac{1}{2} \\ C_{i,\rm ring}(\mathbf{x}, y) &= C_i(\mathbf{x}, y; a_1) - C_i(\mathbf{x}, y; a_2) \\ h^2_{\rm ring}(\mathbf{x}, y) &= \xi_1\, h^2(\mathbf{x}, y; a_1) + \xi_2\, h^2(\mathbf{x}, y; a_2) \end{aligned} \tag{2.49}$$

with a_i the radii of the two circles. Notice that by construction $z_{\rm ring}$ is still $\pm 1/2$ over the $y = 0$ plane. The different signs in z arise from the different orientations of the two boundaries. The integral h^2 is instead independent of the boundary orientation in agreement with the fact that it must be a positive-definite quantity.

The bubbling prescription adopted here is minimal and we don't exclude other interesting choices. However, as we will see in the next section, we reproduce all (to our knowledge) 1/2-BPS D1D5 solutions previously known in the literature. Specifically, we are proposing that *the string profile specifying the solution in the so-called FP representation, gets identified with the boundary of a droplet distribution in the dual D1D5 system.*

3. Adding a tensor multiplet

As we mentioned in the introduction chiral primaries with $h = \bar{h} = j = \bar{j}$ appear only in the KK towers descending from the gravity and tensor multiplets of $\mathcal{N} = (1,0)$ supergravity. In this section we consider 1/2-BPS geometries involving a non-trivial scalar and anti-self-dual three-form in a tensor multiplet dressing the minimal supergravity. They are associated to D1D5 geometries (or any of its dual descriptions).

Instead of starting from some ansatz and apply the logic of section 2.1 to non-minimal supergravity, we jump at the final result, i.e. we adopt our bubbling prescription, and check that the results reproduce regular solutions. In fact, as it was realized in section 2.3, the 1/2-BPS solutions we are interested in, will lie within the general class of solutions for a regular distribution of D1D5 branes [13, 14]. These solutions are specified by a profile function $\mathbf{F}(v)$ determining six harmonic functions[9] f_1, f_5 and C_i in $\mathbb{R}^4$ [14]

$$f_5 = \int_{\partial\mathcal{D}} \frac{dv}{|\mathbf{x}-\mathbf{x}'|^2 + y^2}\,, \qquad f_1 = \int_{\partial\mathcal{D}} \frac{|\partial_v \mathbf{x}'|^2\, dv}{|\mathbf{x}-\mathbf{x}'|^2 + y^2}\,, \quad C_i = \frac{\epsilon_{ij}}{2\pi}\int_{\partial\mathcal{D}} \frac{\partial_v x_i'(v)\, dv}{|\mathbf{x}-\mathbf{x}'|^2 + y^2}\,. \tag{3.1}$$

The metric, three-form flux and scalar profiles are given in terms of f_1, f_5 and C_i via[10]

$$\begin{aligned} ds^2 &= h^{-2}\left[(dt+C)^2 - (d\alpha + B)^2\right] - h^2(dx_1^2 + dx_2^2 + dy^2 + y^2 d\phi^2) && (3.2)\\ G &= d\left[f_1^{-1}(dt+C)\wedge(d\alpha+B)\right] + *_4 df_5 && (3.3)\\ dB &= -*_4\, dC && (3.4)\\ e^{2\Phi} &= f_1 f_5^{-1} \qquad\qquad h^2 = (f_1 f_5)^{\frac{1}{2}}\,. && (3.5) \end{aligned}$$

Recall that we are interested in solutions with $J_{12} = j + \bar{j} = 2j$, $J_{34} = j - \bar{j} = 0$, which correspond to having an additional U(1) isometry — the Killing vector being $\partial/\partial\phi$. We have used this fact to write the ϕ-independent harmonic functions (3.1) in terms of the profile $\mathbf{F}(v) = (\mathbf{x}'(v), 0, 0)$. It then follows from (3.4) that dB is proportional to $d\phi$ and hence we can always write

$$B = z d\phi\,. \tag{3.6}$$

Now, inserting this into (3.4) we reproduce the equation (2.15), namely

$$dC = \frac{1}{y} *_3 dz \tag{3.7}$$

as well as (2.20). After the change of variables (2.39) the metric can be cast in the usual form (2.38) with a (in general) tilted torus fibration.

The supersymmetric solutions are now specified by two harmonic functions in $d = 4$ and one in $d = 6$,namely

$$\Delta_6\left(\frac{z}{y^2}\right) = 0 \tag{3.8}$$

[9]Here we drop "1"s from the harmonic functions, as we are interested in near-horizon geometries. Note that for pure multi-string solutions, with constant profiles, the "1" should be restored.

[10]Notice that the expression for the flux (3.3) corrects a minus sign in the $A\wedge B$ term of the flux given in (2.1) of [14]. We thank O. Lunin for clarifying this point. Then the map is simply $A_{them} = -C_{us}$, and an orientation reversal on the 4d base, $*_{4\,them} = -*_{4\,us}$.

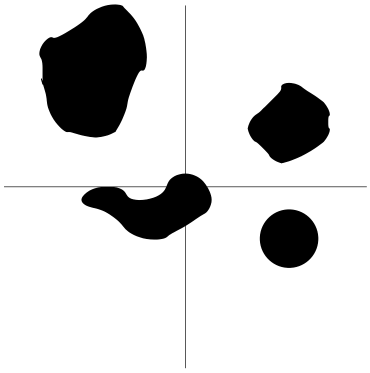

Figure 4: A generic distribution of droplets.

$$\Delta_4 f_i = 0 \qquad i = 1, 5\,. \tag{3.9}$$

Note that in [14] the authors show that if the harmonic functions f_1, f_5, C_i are chosen as in (3.1), with a generic profile, the solution is *non-singular*. This can be used to turn the logic around, and show, using (3.8) and the left hand side of (2.22), that for all non-singular profiles specified by (3.1) the function z must indeed be patch-wise $z = \pm 1/2$ on the $y = 0$ plane. It would be interesting to derive this directly from an analysis of the metric singularities like in [2], and and particular to check whether here more general values of z are allowed.

One can also check that the self-dual three-form $e^{\Phi} G^{+}$ can be written as in (2.44) and the anti-self-dual three-form is given by

$$e^{\Phi} G^{-} = h^2 *_4 d\Phi + e^{+} \wedge e^{-} \wedge d\Phi \tag{3.10}$$

in agreement with the result of [22]. Note that $G = G^{+} + G^{-}$ is of the form

$$G = F_1 \wedge d\theta_1 + F_2 \wedge d\theta_2\,. \tag{3.11}$$

Notice that in the extended supergravity, figures with non-constant velocity $|\partial_v \mathbf{x}'|^2 \neq$const. are allowed. This corresponds to ripplings and irregular droplets that are not guaranteed to be regular solutions in the minimal supergravity. More precisely, the domain $\mathcal{D}$ in (3.1) specifying the solutions is given by a generic distribution of droplets of any shape, see figure 4.

As a simple illustration, let us consider giant gravitons in $d = 6$ dimensions [14]. The corresponding droplet configurations are depicted in 5. They arise from superposing a filled circle (AdS) and a point (a string). The corresponding profile of the constituent solutions are [13]

$$\begin{aligned} \mathbf{x}'(v) &= \left(a \cos\frac{v}{\xi}, a \sin\frac{v}{\xi}\right) \qquad 0 < v < 2\pi\xi \\ \mathbf{x}'(v) &= \mathbf{b} \qquad 0 < v < 2\pi(1-\xi) \end{aligned} \tag{3.12}$$

with $\mathbf{b}$ a constant vector describing the position of the point. The point represents a giant graviton extending in AdS_3 or S^3 depending on whether $|\mathbf{b}|$ is larger or smaller than a [14] — see figure 5.

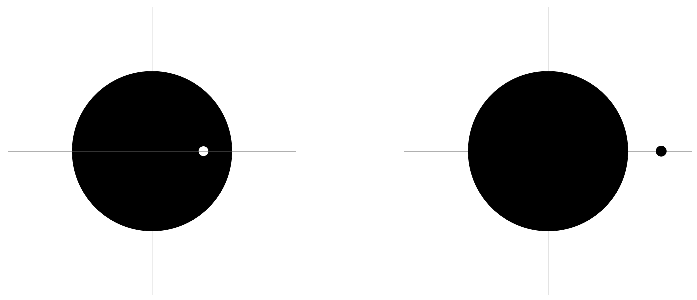

Figure 5: The giant gravitons in [13].

Let us conclude with a final comment. One could wonder whether in the non-minimal supergravity there exist interesting "bubbling" solutions in the case of a rectangular torus. Although this is not expected, we have explicitly checked that this does not happen. All supersymmetric solutions of six dimensional supergravity with a tensor multiplet were analysed[11] in [22], thus extending the results of [10]. Using the results of [22] it is straightforward to repeat the analysis of section 2.1 in the case of non-zero scalar field Φ and an unconstrained flux of the generic form (3.11). The analysis of the supersymmetry conditions then goes through, essentially because the self-dual and anti-self-dual parts of G do not mix [22]. In particular it follows that the metric is exactly as in (2.13). Moreover, equations (2.14), (2.15), (2.15), still hold true, so that z obeys again (2.20). The only difference arises from the flux and its Bianchi identity reproduces the harmonic equations (3.9). However, h^2 is again related to z by (2.19), demonstrating that simply adding a tensors multiplet, but retaining a rectangular torus, *does not* restore linearity of the equations.

4. Concluding remarks

In this note we investigated 1/2-BPS classical geometries arising from bubblings of $AdS_3 \times S^3$ and its pp-wave. This is the supergravity dual of the CFT arising in the IR on the D1D5 system, and 1/2-BPS deformations of $AdS_3 \times S^3$ correspond to chiral primaries in the CFT. We have shown that if one naively mimics the logic of LLM, the resulting set of solutions share most of the properties of their higher dimensional analogs. In particular, rather surprisingly, they are again uniquely specified by a function z which takes values $\pm\frac{1}{2}$ on a two-dimensional plane. However, linearity is lost, due to the fact that the Bianchi identity is not any more automatically satisfied. This leaves room for few possibilities: $AdS_3 \times S^3$, pp-wave and the multi-string solution.

By mapping these solutions to the general form of [10], it becomes clear how one can generalize the ansatz, in order to restore linearity. This is accomplished by allowing for a generically tilted torus with $\mathrm{SO}(2) \times \mathrm{SO}(2)$ isometries. The upshot of this is that 1/2-BPS solutions dual to chiral primaries operators with $h = \bar{h} = j = \bar{j}$ are now specified by z and an additional function h^2 harmonic in $\mathbb{R}^4$. The functions z and h^2 are written in terms

[11]Note that the authors of [22] considered the more general case of gauged supergravities. However, for our purposes the gauge fields are set to zero.

of integrals over the one-dimensional boundary of a droplet distribution with $z = \pm 1/2$ in the $y = 0$ plane, i.e. harmonics generated by lines of charges. The geometries display an interesting fibration of the symmetry torus over a four-dimensional base, with a special two-dimensional plane where one-cycles shrink to zero size, while keeping the whole geometry regular.

After adding a tensor multiplet a wide variety of previously known D1D5 classical geometries are reproduced. For instance, giant gravitons are reinterpreted as superposition of points over AdS disks. Resuming, bubbling works much as in the ten-dimensional case, but droplet boundaries reveal to be more fundamental than the droplets themselves.

Perhaps this is not very surprising. In fact, the supergravity duals of chiral primaries of the D1D5 system were constructed in [13, 14], in terms of a profile function $\mathbf{x}'(v)$ arising via a chain of dualities from fundamental string-momentum solutions [23, 24, 25]. Here we identified the profile with the boundary of the droplets. The results are consistent with a free fermion description of the chiral primaries in the two-dimensional CFT [11] and it would be nice to make this correspondence precise. Notice that our analysis showed that the Killing spinors are charged under the two $\mathrm{U}(1) \times \mathrm{U}(1)$ isometry and therefore fermions can get masses even if the internal space is flat.

Acknowledgments

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A. Derivation of the solutions for a rectangular torus

In this appendix we present the detailed derivation of the solution summarized in 2.2. We start with the following ansatz for the six-dimensional metric and the three-form:

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, dx^\mu dx^\nu - e^{2A}\, d\theta_1^2 - e^{2D}\, d\theta_2^2 \\ G &= F \wedge d\theta_1 + \tilde{F} \wedge d\theta_2 \,. \end{aligned} \tag{A.1}$$

We utilize the standard technique of analysing the supersymmetry conditions encoded in a set of form bi-linears [6]. The tensors we consider are the timelike vector K and the forms defined in (2.12), which are related to each other (using self–duality of X^i) as:

$$\begin{aligned} \tilde{Y}^i &= -e^{-A+D} *_4 Y^i\,, \qquad & Y^i &= e^{A-D} *_4 \tilde{Y}^i \\ \tilde{L}^i &= e^{-A-D} *_4 L^i\,, \qquad & L^i &= -e^{A+D} *_4 \tilde{L}^i\,. \end{aligned} \tag{A.2}$$

Using the algebraic relations (2.2) and equation (2.12) of [10], we obtain the following relations:

$$K \cdot K = e^{2A} + e^{2D} \tag{A.3}$$

$$L^i \cdot L^j = -\delta^{ij} e^{A+D} K \cdot K\,, \tag{A.4}$$

$$
\begin{aligned}
K \cdot L^i &= 0 && \text{(A.5)} \\
L^{i\mu} Y^j_{\mu\nu} &= -\epsilon^{ijk} e^{2A} L^k_\nu - \delta^{ij} K_\nu e^{2(A+D)} \,. && \text{(A.6)}
\end{aligned}
$$

It follows that we can use (K, L^i), appropriately normalized, as a privileged orthonormal frame in four dimensions. Moreover, using (A.4) with (A.6) one can get explicit expressions for Y^i and $\tilde{Y}^i$, in terms of K and L^i:

$$
\begin{aligned}
Y^i &= -\frac{1}{2} e^{-2D} (K \cdot K)^{-1} \epsilon^{ijk} L^j \wedge L^k - (K \cdot K)^{-1} K \wedge L^i && \text{(A.7)} \\
\tilde{Y}^i &= -\frac{1}{2} e^{-2A} (K \cdot K)^{-1} \epsilon^{ijk} L^j \wedge L^k + (K \cdot K)^{-1} K \wedge L^i \,. && \text{(A.8)}
\end{aligned}
$$

A.1 Analysis of the supersymmetry conditions

We now turn to the differential conditions. The antisymmetric part of (2.3) gives

$$
\begin{aligned}
dK &= 2(F + \tilde{F}) && \text{(A.9)} \\
de^{2A} &= -2 i_K F && \text{(A.10)} \\
de^{2D} &= -2 i_K \tilde{F} \,. && \text{(A.11)}
\end{aligned}
$$

The differential conditions on X^i read

$$
\begin{aligned}
d\tilde{L}^i &= 0 \\
dL^i &= \partial_{\theta_2} Y^i - \partial_{\theta_1} \tilde{Y}^i \\
dY^i &= \partial_{\theta_1} \tilde{L}^i \\
d\tilde{Y}^i &= \partial_{\theta_2} \tilde{L}^i \,. && \text{(A.12)}
\end{aligned}
$$

Notice that the forms generically depend on θ_1, θ_2, reflecting the fact that the Killing spinors are "charged" under the corresponding U(1) isometries. Indeed, it can be shown that if one assumes that the forms do not depend on the angular variables on $S^1 \times S^1$, the system does not have non-trivial solutions.

We now solve this set of equations and find the general supersymmetric background preserving (2.5). After some algebra, the system (A.12) is shown to be equivalent to the following set of conditions

$$
\begin{aligned}
dL^i &= \frac{1}{2} e^{-2(A+D)} \partial_{\theta_1} \left(\epsilon^{ijk} L^j \wedge L^k \right) && \text{(A.13)} \\
d\left(*_3 L^i \right) &= -e^{A+D} h d \left(\frac{1}{h e^{A+D}} \right) \wedge *_3 L^i && \text{(A.14)} \\
*_3 \, \partial_{\theta_1} L^i &= h \, e^{A+D} dL^i \,, && \text{(A.15)} \\
*_3 \, \partial_{\theta_2} L^i &= -h \, e^{A+D} dL^i && \text{(A.16)} \\
dC \wedge L^i &= \frac{1}{2} d \left(e^{-2A} h^2 \epsilon^{ijk} L^j \wedge L^k \right) && \text{(A.17)} \\
dC \wedge L^i &= -\frac{1}{2} d \left(e^{-2D} h^2 \epsilon^{ijk} L^j \wedge L^k \right) . && \text{(A.18)}
\end{aligned}
$$

Compatibility of (A.15) and (A.16) shows that $\partial_{\theta_1} L^i = -\partial_{\theta_2} L^i$. The equations (A.17) and (A.18) instead imply that

$$d\left(e^{-2(A+D)}\epsilon^{ijk} L^j \wedge L^k\right) = 0\,. \tag{A.19}$$

A more useful expression which we will use to solve the above conditions follows from the $(\mu\nu\theta_1\theta_2)$ component of (2.4):

$$\nabla_\mu L^i_\nu = \partial_\mu (A+D) L^i_\nu + F_\mu{}^\rho Y^i_{\rho\nu} + F_\nu{}^\rho Y^i_{\rho\nu} + \frac{1}{2} g_{\mu\nu} F_{\rho\lambda} Y^{i\ \rho\lambda}\,. \tag{A.20}$$

The antisymmetric part of (A.20) gives

$$d(e^{-(A+D)} L^i) = 0\,, \tag{A.21}$$

whose general solution is given by

$$L^i = e^{(A+D)} R^i_j(\theta_1, \theta_2) dx^j\,. \tag{A.22}$$

It can be shown that the matrix R must be an SO(3) rotation, and using this, together with the relation (A.4) allows us to read off the complete form of the metric, which we write below:

$$ds^2 = h^{-2}(dt+C)^2 - h^2\left(dx_1^2 + dx_2^2 + dx_3^2\right) - e^{2A}\, d\theta_1^2 - e^{2D}\, d\theta_2^2\,. \tag{A.23}$$

This now tells us that

$$*_3\, L^i = \frac{1}{2} e^{-(A+D)} h\, \epsilon^{ijk} L^j \wedge L^k\,, \tag{A.24}$$

and we can solve the constraints (A.14)–(A.13). The first condition (A.14) is identically satisfied. Eq. (A.15) determines now the θ_i dependence of L^i. After some algebra we get

$$\partial_{\theta_1} R^i_j = R^i_l \epsilon_{jkl} \partial_l e^{A+D}\,. \tag{A.25}$$

Since R is an SO(3) matrix, it follows that one of the three x^i coordinates is

$$x^3 = y = e^{A+D} \tag{A.26}$$

and we define $e^G = e^{A-D}$. In this way, R must be a rotation in the other two coordinates by an angle $\theta_1 - \theta_2$, so that we also solve (A.16). (A.13) is now trivially satisfied. Next we solve (A.9) and (A.10)–(A.11). Using the explicit form of K as a form, (A.9) reads

$$d(h^{-2}(dt+C)) = 2(F + \tilde{F})\,. \tag{A.27}$$

Following [2], we pose

$$B = B_t(dt+C) + \hat{B} \tag{A.28}$$

$$\tilde{B} = \tilde{B}_t(dt+C) + \hat{\tilde{B}}\,, \tag{A.29}$$

hence (A.10) and (A.11) give

$$dB_t = \frac{1}{2} d(y\, e^{G})\,, \qquad d\tilde{B}_t = \frac{1}{2} d(y\, e^{-G})\,, \tag{A.30}$$

which can be integrated to

$$B_t = \frac{y}{2}\, e^{G}\,, \qquad \tilde{B}_t = \frac{y}{2} e^{-G}\,, \tag{A.31}$$

and we have set to zero irrelevant integration constants. Inserting these values into (A.27), one component is identically satisfied using (2.8) while the non–trivial part implies

$$d\hat{B} + d\hat{\tilde{B}} = 0\,. \tag{A.32}$$

Now consider the 3–form flux. Self-duality implies:

$$d\hat{B} + B_t dC \;=\; h^2 e^{G} \, *_3 \, d\tilde{B}_t \tag{A.33}$$

$$d\hat{\tilde{B}} + \tilde{B}_t dC \;=\; -h^2 e^{-G} \, *_3 \, dB_t\,. \tag{A.34}$$

Summing the two equations (A.33) and (A.34) one finds

$$dC = 2h^4\, y *_3 dG = \frac{1}{y} *_3\, dz \qquad z = \frac{1}{2}\tanh G\,. \tag{A.35}$$

Notice that this also solves equations (A.17), (A.18). Finally, $d\hat{B}$ can be read off from either of (A.33), (A.34) and reads

$$d\hat{B} = -d\hat{\tilde{B}} = \frac{1}{2} y \, *_3 \, dh^2\,. \tag{A.36}$$

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